Analytic General Solutions of Nonlinear Difference Equations

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Abstract

There is no general existence theorem for solutions for nonlinear difference equations, so we must prove the existence of solutions in accordance with models one by one.

In our work, we found theorems for the existence of analytic solutions of nonlinear second order difference equations. The main work of the present paper is obtaining representations of analytic general solutions with new methods of complex analysis.

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1 Introduction

There is a general existence theorem for solutions of analytic differential equations, but we have no general existence theorem for analytic difference equations. For example, we consider the following first order nonlinear difference equation

$$x(t+1) = 2x(t) + x(t)^2. \tag{$*$}$$

Putting $x(t) = -1 + y(t)$, we get $y(t+1) = y(t)^2$ and $\log y(t+1) = 2 \log y(t)$. Then $u(t) = 2^t$ is a (particular) solution of the equation $u(t+1) = 2u(t)$. Putting $C(t) =$

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$(\log y(t))/u(t)$, we have $C(t+1) = C(t)$, that is $C(t) = \pi(t)$, where $\pi(t)$ is an entire solution with the period 1. Therefore, a general solution of $(*)$, which tends to 0 as $t \to -\infty$, is given by $x(t) = \exp[\pi(t)2^t] - 1$.

This simple example $(*)$ illustrates the whole make-up of the present paper. 0 is its equilibrium point of $(*)$, with the characteristic value 2. A formal solution of it is obtained by putting $x(t) = \sum_{n=1}^{\infty} a_n (2^t)^n$. If its convergence is shown, then we have a solution $x(t)$ of the initial value problem, which tends to 0 as $t \to -\infty$. Further we proceed to seek general solutions.

For analytic differential equations, a solution of initial value problem is always represented by a power series. This is the reason that the general existence theorem can be established for differential equations. But for difference equations, this is not the case. Next we consider the following first order difference equation

$$x(t+1) = x(t) + x(t)^2, \tag{$**$}$$

for which 0 is the equilibrium point with characteristic value 1, but we can not put its formal solution in the form $\sum_{n=1}^{\infty} a_n (1^t)^n$. That is, selection of appropriate formal solution depends on the problem. Of course, by [5] p.237 Theorem 14.2, $(**)$ has a local solution with the asymptotic expansion

$$x(t) \sim -\frac{1}{t} \left\{ 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t} \right)^k \right\}^{-1},$$

where $\hat{q}_{jk}$ are constants.

Further, for analytic differential equations, the solution is determined uniquely by the initial condition. However, for analytic difference equations, solution cannot be determined by the (initial) condition $x(t) \to 0$ as $t \to -\infty$, hence we need to consider general solutions.

In this paper, we consider the following second order nonlinear difference equation,

$$u(t+2) = f(u(t), u(t+1)), \tag{1.1}$$

where $f(x, y)$ is an entire function of x, y. We assume that there is an equilibrium point $u^* : u^* = f(u^*, u^*)$. We can take $u^* = 0$, that is $f(0, 0) = 0$ without losing generality.

Many studies for difference equations are considered with discrete variables. Indeed such the equation (1.1) is often considered for $t \in \mathbb{N}$. However in our study, we consider the difference equation (1.1) with a continuous variable t. If "t" of equation (1.1) represents "time", then t is of course a real variable. However hereafter, in (1.1), t represents a complex variable, because we consist more general theorems.

Our aim is to obtain analytic general solutions $u(t)$ of (1.1) such that $u(t+n) \to 0$ as $n \to +\infty$ or $n \to -\infty$.

We define $f(x, y)$ in (1.1) such that

$$f(x, y) = -\beta x - \alpha y + g(x, y), \quad \beta \neq 0, \tag{1.2}$$

where g consists of higher order terms for x, y such that $g(x,y) = \sum_{i,j \geqq 0, i+j \geqq 2} b_{i,j} x^i y^j \not\equiv 0$, and α, β, $b_{i,j}$ are constants. Further we assume that at least one of moduli of the characteristic values is neither 0 nor 1. The case that both of characteristics equal to 1 will be treated in another paper.

The processes of my work are as follows: **1)** determination of formal solutions, **2)** getting particular solution by Schauder's Fixed Point Theorem in a locally convex topological space, **3)** obtaining general solutions by the method of Kimura[5] and Yanagihara[11].

2 Analytic Solutions

2.1 A formal solution.

The characteristic equation of (1.1) with (1.2) is

$$D(\lambda) = \lambda^2 + \alpha\lambda + \beta = 0. \tag{2.1}$$

Let λ_1, λ_2 be roots of the characteristic equation and $|\lambda_1| \leqq |\lambda_2|$. Then we consider following two cases, i) $|\lambda_1| < 1$ and ii) $|\lambda_2| > 1$. Of course, some characteristic equations have properties both i) and ii).

In case i), we consider solutions such that

$$u(t+n) \to 0, \quad \text{as } n \to +\infty.$$

In case ii), we consider solutions such that

$$u(t+n) \to 0, \quad \text{as } n \to -\infty.$$

In case i) we put $\lambda = \lambda_1$, and in case ii) we put $\lambda = \lambda_2$. Then we put a formal solution to (1.1)

$$u(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt},$$

in both cases. We substitute $u(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}$, $u(t+1) = \sum_{n=1}^{\infty} a_n \lambda^{n(t+1)}$, $u(t+2) = \sum_{n=1}^{\infty} a_n \lambda^{n(t+2)}$, into (1.1). And we compare the coefficients of λ^{nt}, $(n = 1, 2, \cdots)$, then we have, with $D(\lambda)$ in (2.1),

$$\begin{cases} a_1 \cdot D(\lambda) = 0, \\ a_2 \cdot D(\lambda^2) = a_1^2 (b_{2,0} + b_{1,1}\lambda + b_{0,2}\lambda^2), \\ a_3 \cdot D(\lambda^3) = b_{2,0} 2a_1 a_2 + b_{1,1} a_1 a_2 \lambda(\lambda+1) + b_{0,2} 2a_1 a_2 \lambda^3 \\ \qquad\qquad\qquad\qquad\qquad\qquad + a_1^3 (b_{3,0} + b_{2,1}\lambda + b_{1,2}\lambda^2 + b_{0,3}\lambda^3), \\ \cdots \\ a_k \cdot D(\lambda^k) = C_k(a_1, \cdots, a_{k-1}), \\ \cdots, \end{cases}$$

where $C_k(a_1, \cdots, a_{k-1})$ are polynomials of $a_1, \cdots, a_{k-1}$ with coefficients $b_{i,j}\lambda^l$, $0 \leqq i \leqq k$, $0 \leqq j \leqq k$, $0 \leqq l \leqq k$, $2 \leqq i+j \leqq k$. From definition of λ and D, we have $D(\lambda) = 0$ and $D(\lambda^k) \neq 0\,(k \geqq 2)$, and we can have a_1 is arbitrary.

Here we suppose that $a_1 \neq 0$. Then we have

$$a_k = \frac{a_1^k}{D(\lambda^k)} C_k^*(b_{i,j}, \lambda^l), \;\; k \geqq 2, \tag{2.2}$$

where $C_k^*(b_{i,j}, \lambda^l)$ are constants which are given by the function f, in which they consist of $b_{i,j}$, $2 \leqq i+j \leqq k$ and λ^l, $0 \leqq l \leqq k$. Hence we can determine a formal solution of (1.1),

$$u(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}, \tag{2.3}$$

in both cases i) and ii). Here we have a_1 is arbitrary not 0, and a_k are determined by a_1.

2.2 Existence of an analytic solution

Here we put $u(t) = s, u(t+1) = w, u(t+2) = z$, and $H(s,w,z) = -z + f(s,w)$. Then the equation (1.1) can be written such as

$$H(u(t), u(t+1), u(t+2)) = 0. \tag{2.4}$$

$H(s,w,z)$ is holomorphic in a neighborhood of $(0,0,0)$ and we have $H(0,0,0) = 0$ easily. Furthermore we have $\frac{\partial H}{\partial s}(0,0,0) = \frac{\partial f}{\partial s}\Big|_{s=w=0} = -\beta \neq 0$ as remarked in (1.2). From implicit function theorem, for the equation $H(s,w,z) = 0$, we have a holomorphic function ϕ such that

$$s = \phi(w,z) \quad \text{for} \quad |w|,\, |z| \leqq \rho \tag{2.5}$$

for some $\rho > 0$. Furthermore we have a constant K such that

$$|s| = |\phi(w,z)| \leqq K(|w| + |z|) \quad \text{for} \quad |w|, |z| \leqq \rho. \tag{2.6}$$

Let N be a positive integer. Put the partial sum of formal solution as $P_N(t) = \sum_{n=1}^{N} \alpha_n \lambda^{nt}$, and put $p_N(t) = u(t) - P_N(t)$. Here we rewrite $p(t) = p_N(t)$.

Moreover we define following sets,

$$\begin{aligned} &S(\eta) = \{t \in \mathbb{C} : |\lambda^t| \leqq \eta\}, \\ &J(A, \eta) = \{p : p(t) \,\text{is holomorphic and}\; |p(t)| \leqq A|\lambda^t|^{N+1} \,\text{for}\; t \in S(\eta)\}. \end{aligned}$$

in which $A > 0$ and η, $0 < \eta < 1$ are constants. We determined these constants in a proof of existence for a fixed point of following maps T_i $(i = 1, 2)$.

Suppose there would exist a solution $u(t)$ of (1.1) in $S(\eta)$. Then $p_N(t) = u(t) - P_N(t)$ would belong to $J(A, \eta)$ for some suitably chosen constants A, η, and would satisfy the equation

$$p(t+2) = f(p(t) + P_N(t), p(t+1) + P_N(t+1)) - P_N(t+2), \tag{2.7}$$

with $p(t) = p_N(t)$. Conversely, suppose there would exist a solution $p(t)$ of (2.7), then $u(t) = p(t) + P_N(t)$ would be a solution of (1.1). So, hereafter we concentrate on proving the existence of $p(t) \in J(A, \eta)$ such that $u(t) = p(t) + P_N(t)$ satisfies (2.7).

In case i) $|\lambda| < 1$, the existence of solutions $u(t)$ of (2.7) is equivalent to the existence of $p(t)$ which satisfies

$$p(t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t).$$

For $p(t) \in J(A, \eta)$, we put

$$T_1[p](t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t). \tag{2.8}$$

Then we can prove that T_1 maps $J(A, \eta)$ into itself (see Appendix A). The map T_1 is obviously continuous if $J(A, \eta)$ is endowed with topology of uniform convergence on compact sets in $S(\eta)$. Furthermore $J(A, \eta)$ is clearly convex, and is relatively compact by the theorem of Montel [1].

By Schauder's fixed point theorem [2](p.74), [6](p.32), we obtain the existence of a fixed point $p(t) = p_N(t) \in J(A, \eta)$ of T_1 in $S(\eta)$. Moreover we can prove uniqueness of the fixed point (see Appendix B) and independence from N (see Appendix C). Hence we have an analytic solution $u(t)$ in $S(\eta)$.

In case ii) $|\lambda| > 1$, (2.7) is equivalent to the existence of $p(t)$ which satisfies,

$$p(t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t).$$

For $p(t) \in J(A, \eta)$, we put

$$T_2[p](t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t)$$

Then we can prove the existence of an analytic solution $u(t)$ in $S(\eta)$ by the arguments similar as above.

Thus we have the following Theorem 1.

Theorem 1. *Let λ_1, λ_2 be roots of $D(\lambda) = 0$ in (2.1), with $|\lambda_1| \leqq |\lambda_2|$. Suppose $|\lambda_1| < 1$ or $|\lambda_2| > 1$. Put $\lambda = \lambda_1$ for the former, and $\lambda = \lambda_2$ for latter. And we assume that $\lambda_1^k \neq \lambda_2$ and $\lambda_2^k \neq \lambda_1$ for any $k \in \mathbb{N}$. Then there is an $\eta > 0$ such that we have a holomorphic solution $u(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}$ in $S(\eta) = \{t; |\lambda^t| < \eta\}$.*

In case ii). The solution $u(t)$ can be analytically continued to the whole plane, by making use of the equation (1.1) $u(t+2) = f(u(t), u(t+1))$.

In case i). The function $\phi(w,z)$ in (2.5) $s = \phi(w,z)$ for $|w|$, $|z| \leqq \rho$ is defined only locally, though we can also analytically continue $u(t)$, keeping out of branch points. The solution obtained is multi-valued.

2.3 Particular solutions.

In this subsection, we consider solutions $u_1(t)$ and $u_2(t)$ which are respectively depend on λ_1 and λ_2. Put formal solutions such that $u_1(t) = \sum_{n=1}^{\infty} a_{1,n}\lambda_1^{nt}$ and $u_2(t) = \sum_{n=1}^{\infty} a_{2,n}\lambda_2^{nt}$, we have $a_{m,k} \cdot D(\lambda_m^k) = C_{m,k}(a_{m,1}, \cdots, a_{m,k-1})$, $(m = 1, 2; k \in \mathbb{N})$ with the similar arguments in subsection **2.1**, where $C_{m,k}(a_{m,1}, \cdots, a_{m,k-1})$ are polynomials of $a_{m,1}, \cdots, a_{m,k-1}$ with coefficients $b_{i,j}\lambda_m^l$, $0 \leqq i \leqq k$, $0 \leqq j \leqq k$, $0 \leqq l \leqq k$, $2 \leqq i+j \leqq k$. Furthermore if we take as $a_{m,1} \neq 0$, then we have

$$a_{m,k}D(\lambda_m^k) = a_{m,1}^k C_{m,k}^*(b_{i,j}, \lambda_m^l), \ m = 1, 2; \ k \geqq 2, \tag{2.9}$$

where $C_{m,k}^*(b_{i,j}, \lambda_m^l)$ are constants which are given by the function f, in which they consist of $b_{i,j}$, $2 \leqq i+j \leqq k$ and λ_m^l, $0 \leqq l \leqq k$. Then we have the following lemma 2 and lemma 3 with the similar arguments in **2.1-2.2**.

Lemma 2. *Let λ_1, λ_2 be roots of (2.1) with $|\lambda_1| \leqq |\lambda_2| < 1$. If $\lambda_2^k \neq \lambda_1$ for any positive integer k greater than 1, then there are constants η_1, $\eta_2 > 0$ such that we have following two holomorphic solutions u_1 and u_2 of (1.1),*

$$u_m(t) = \sum_{n=1}^{\infty} a_{m,n}\lambda_m^{nt} \quad in \quad S(\eta_m) = \{t; |\lambda_m^t| < \eta_m\}, \quad (m = 1, 2),$$

in which $a_{1,1}$ and $a_{2,1}$ can be taken to be arbitrary non-zero constants.

For the case $\lambda_2^k = \lambda_1$ for some $k \in \mathbb{N}$, if $C_{2,k}^*(b_{i,j}, \lambda_2^l) = 0$ given in (2.9), then we take $a_{2,1} \neq 0$ and $a_{2,k} \neq 0$ arbitrary, and have the solution $u_2(t)$ as above. On the other hand, if $C_{2,k}^*(b_{i,j}, \lambda_2^l) \neq 0$ for the k, then we take $a_{2,j} = 0$ for $(j \neq kn, n \in \mathbb{N})$, and can take $a_{2,k} \neq 0$ arbitrary, then we determine coefficients $a_{2,kn}$ for $n \geqq 2$ as above. Hence then there is an $\eta_2 > 0$ such that we have holomorphic solutions u_1 and u_2 of (1.1),

$$u_2(t) = \sum_{n=1}^{\infty} a_{2,kn}\lambda_2^{knt} \quad in \ S(\eta_2),$$

as well as $u_1(t) = \sum_{n=1}^{\infty} a_{1,n}\lambda_1^{nt}$ in $S(\eta_1)$. In the case of $\lambda_2^k = \lambda_1$ and $C_{2,k}^*(b_{i,j}, \lambda_2^l) \neq 0$, if we take $a_{2,k} = a_{1,1}$, then $u_2(t) = u_1(t)$ in $S(\eta_1) \cap S(\eta_2)$.

Thus, in the both cases, $u_1(t+n) \to 0, u_2(t+n) \to 0$ as $n \to \infty$ uniformly on any compact subset of the t-plane.

Proof. If $\lambda_2^k \neq \lambda_1$ for any $k \in \mathbb{N}$, then we can determine formal solution $u_2(t) = \sum_{n=1}^{\infty} a_{2,n}\lambda_2^{nt}$ as in subsection **2.1**, with $\lambda = \lambda_2$ instead of λ_1. And we can show that it is an actual solution as in subsections **2.1-2.2**

For the case $\lambda_2^k = \lambda_1$ for some $k \in \mathbb{N}$, if we take $a_{2,1} \neq 0$, form (2.9), we have

$$a_{2,k}D(\lambda_2^k) = a_{2,k}D(\lambda_1) = a_{2,1}^k C_{2,k}^*(b_{i,j}, \lambda_2^l) = 0. \tag{2.10}$$

If $C_{2,k}^*(b_{i,j}, \lambda_2^l) = 0$, we can take $a_{2,k}$ arbitrary, and determine $a_{2,n}$, $2 \leqq n \leqq k-1, n \geqq k+1$ by $a_{2,1}$ as in (2.9).

However if $C_{2,k}^*(b_{i,j}, \lambda_2^l) \neq 0$, the equation (2.10) is contradiction. Thus we must take $a_{2,1} = 0$, then $a_{2,n} = 0$, for $n \leqq k-1$ by $a_{2,k} \cdot D(\lambda_2^k) = C_{2,k}(a_{2,1}, \cdots, a_{2,k-1})$. Then we can take $a_{2,k}$ to be arbitrary non-zero constant, and determine coefficients $a_{2,n}$ as follows

$$a_{2,n} = \begin{cases} 0, \quad (n \neq km,\ m \in \mathbb{N}), \\ a_{2,k}^m \frac{C_{2,m}^*(b_{i,j}, \lambda_2^{lk})}{D(\lambda_2^{km})} = a_{2,k}^m \frac{C_{2,m}^*(b_{i,j}, \lambda_1^l)}{D(\lambda_1^m)}, \quad (n = km,\ m \in \mathbb{N}), \end{cases}$$

where $C_{2,m}^*(b_{i,j}, \lambda_1^l)$ are constants defined in (2.9). Hence we can determine a formal solution $u_2(t)$ such that

$$u_2(t) = \sum_{n=1}^{\infty} a_{2,kn}\lambda_2^{knt} \quad \text{in } S(\eta_2).$$

If we take $a_{2,k} = a_{1,1}$, then we have only one solution. Futhermore for the both cases of $\lambda_2^k = \lambda_1$, we can prove that there is an $\eta_2 > 0$ such that we have a holomorphic solution $u_2 = \sum_{n=1}^{\infty} a_{2,kn}\lambda_2^{knt}$ in $S(\eta_2)$, with the similar arguments in **2.2**. in $S(\eta_1) \cap S(\eta_2)$.

Obviously $u_1(t+n) \to 0, u_2(t+n) \to 0$ as $n \to +\infty$ uniformly on any compact subset of the t-plane. $\square$

Lemma 3. *Let λ_1, λ_2 be roots of (2.1) with $1 < |\lambda_1| \leqq |\lambda_2|$. If $\lambda_1^k \neq \lambda_2$ for any positive integer k greater than 1, then there are constants $\eta_1, \eta_2 > 0$ such that we have following two holomorphic solutions u_1 and u_2 of (1.1),*

$$u_m(t) = \sum_{n=1}^{\infty} a_{m,n}\lambda_m^{nt} \quad in \quad S(\eta_m) = \{t; |\lambda_m^t| < \eta_m\}, \quad (m = 1, 2),$$

in which $a_{1,1}$ and $a_{2,1}$ can be taken to be arbitrary non-zero constants.

For the case $\lambda_1^k = \lambda_2$ for some $k \in \mathbb{N}$, if $C_{1,k}^*(b_{i,j}, \lambda_1^l) = 0$ given in (2.9), then we take $a_{1,1} \neq 0$ and $a_{1,k} \neq 0$ arbitrary, and have the solution $u_1(t)$ as above. On the other hand, if $C_{1,k}^*(b_{i,j}, \lambda_1^l) \neq 0$ for the k, then we take $a_{1,j} = 0$ for $(j \neq kn, n \in \mathbb{N})$, and can $a_{1,k} \neq 0$ arbitrary, then we determine coefficients $a_{1,kn}$ for $n \geqq 2$ as above. Hence then there is an $\eta_1 > 0$ such that we have holomorphic solutions u_1 and u_2 of (1.1),

$$u_1(t) = \sum_{n=1}^{\infty} a_{1,kn}\lambda_1^{knt} \quad in\ S(\eta_1),$$

as well as $u_2(t) = \sum_{n=1}^{\infty} a_{2,n}\lambda_2^{nt}$ *in* $S(\eta_2)$. *In the case of* $\lambda_1^k = \lambda_2$ *and* $C^*_{1,k}(b_{i,j}, \lambda_1^l) \neq 0$, *if we take* $a_{1,k} = a_{2,1}$, *then* $u_1(t) = u_2(t)$ *in* $S(\eta_1) \cap S(\eta_2)$.

Thus, in the both cases, $u_1(t-n) \to 0, u_2(t-n) \to 0$ *as* $n \to \infty$ *uniformly on any compact subset of the* t*-plane.*

Proof. We can prove with arguments similar as Lemma 2. $\square$

The analytic solutions u_1 and u_2 obtained in Lemmas 2-3 are "Particular Solutions" of (1.1).

3 Analytic General Solutions

Analytic general solutions of nonlinear difference equations have been investigated, for example, by Harris [3], [4], and others, but we can not make use of their method. Here we follow the method of Kimura [5] and Yanagihara [11], where general solutions of the first order difference equations are studied.

In this section we consider the following case,

$$|\lambda_1| < 1 < |\lambda_2|.$$

For other cases, we study general solutions of the difference equation (1.1) in other papers.

For a linear second order difference equation, general solutions are written by two particular solutions of it. But for a nonlinear second order difference equation, in this case, general solutions which converge to an equilibrium point of the equation are written by one of two particular solutions u_1 or u_2 of the difference equation.

Let $u(t)$ be a solution of (1.1), and $w(t) = u(t+1)$. Then (1.1) can be written as a system of simultaneous equations

$$\begin{pmatrix} u(t+1) \\ w(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \begin{pmatrix} u(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 \\ g(u(t), w(t)) \end{pmatrix} \tag{3.1}$$

Let λ_1, λ_2 be roots of the equation (2.1) and $P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$. Put

$$\begin{pmatrix} u \\ w \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.2}$$

From $\lambda_1 \neq \lambda_2$, we can transform the coefficient matrix of linear terms of (3.1) into

diagonal form, i.e., (3.1) is transformed to a following system with respect to x, y :

$$\begin{cases} x(t+1) = \lambda_1 x(t) + \sum_{i+j\geq 2} c_{ij} x(t)^i y(t)^j = X(x(t), y(t)), \\ y(t+1) = \lambda_2 y(t) + \sum_{i+j\geq 2} d_{ij} x(t)^i y(t)^j = Y(x(t), y(t)). \end{cases} \tag{3.3}$$

On the other hand, let $Q = \begin{pmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{pmatrix}$. Put

$$\begin{pmatrix} u \\ w \end{pmatrix} = Q \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.4}$$

Then (3.1) is transformed to a system with respect to x, y :

$$\begin{cases} x(t+1) = \lambda_2 x(t) + \sum_{i+j\geq 2} c'_{ij} x(t)^i y(t)^j = X'(x(t), y(t)), \\ y(t+1) = \lambda_1 y(t) + \sum_{i+j\geq 2} d'_{ij} x(t)^i y(t)^j = Y'(x(t), y(t)). \end{cases} \tag{3.5}$$

Then we will show the following Theorem 4.

Theorem 4. *Let λ_1, λ_2 be roots of the characteristic equation of (1.1) such that $|\lambda_1| < 1 < |\lambda_2|$. Suppose that $u_1(t)$ and $u_2(t)$ are solutions of (1.1) which have the expansions $u_1(t) = \sum_{n=1}^{\infty} a_{1,n}\lambda_1^{nt}$ in $S(\eta_1) = \{t; |\lambda^t| < \eta_1\}$, $u_2(t) = \sum_{n=1}^{\infty} a_{2,n}\lambda_2^{nt}$ in $S(\eta_2) = \{t; |\lambda^t| < \eta_2\}$ with some constants $\eta_1, \eta_2 > 0$. Further suppose that $\Upsilon(t)$ is an analytic solution of (1.1) such that either $\Upsilon(t+n) \to 0$ as $n \to +\infty$ or $n \to -\infty$, uniformly on any compact subsets of t-plane. If the solution Υ of (1.1) satisfies $\Upsilon(t + (-1)^{m-1}n) \to 0$, $(m = 1, 2)$ as $n \to +\infty$, then there is a periodic entire function $\pi_m(t), (\pi_m(t+1) = \pi_m(t))$, such that*

$$\begin{aligned} \Upsilon(t) = {} & \frac{1}{\lambda_{m+1} - \lambda_m}(\lambda_{m+1} \sum_{n=1}^{\infty} a_{m,n} \lambda_m^{n(t+\pi_m(t))} - \sum_{n=1}^{\infty} a_{m,n} \lambda_m^{n(t+\pi_m(t)+1)}) \\ & + \Psi_m \left(\frac{1}{\lambda_{m+1} - \lambda_m}(\lambda_{m+1} \sum_{n=1}^{\infty} a_{m,n} \lambda_m^{n(t+\pi_m(t))} - \sum_{n=1}^{\infty} a_{m,n} \lambda_m^{n(t+\pi_m(t)+1)}) \right), \end{aligned} \tag{3.6}$$

in $S(\eta_m)$, with the convention λ_3 means λ_1. Further we have $\frac{\Upsilon(t+1+(-1)^{m-1}n)}{\Upsilon(t+(-1)^{m-1}n)} \to \lambda_m$, $(m = 1, 2)$, as $n \to +\infty$.

When $m = 1$, Ψ_1 is a solution of

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{3.7}$$

and when $m = 2$, Ψ_2 is a solution of

$$\Psi(X'(x, \Psi(x))) = Y'(x, \Psi(x)), \tag{3.8}$$

in which X, Y are defined in (3.3), and X', Y' are defined in (3.5).

Conversely, a function $\Upsilon(t)$ which is represented as in (3.6) in $S(\eta_m)$ for some $\eta_m > 0$, where $\pi_m(t)$ is a periodic function with the period one, is a solution of (1.1) such that $\Upsilon(t + (-1)^{m-1}n) \to 0$ and $\frac{\Upsilon(t+1+(-1)^{m-1}n)}{\Upsilon(t+(-1)^{m-1}n)} \to \lambda_m$ as $n \to +\infty$ with $m = 1, 2$.

Proof. At first we prove the case $m = 1$.

Let $u(t)$ be the solution of (1.1) in the argument of Section 2. And suppose $\Upsilon(t)$ be a solution of (1.1) such that $\Upsilon(t + n) \to 0$ as $n \to +\infty$ uniformly on any compact subsets of t-plane.

At first we will consider the meaning of the functional equation (3.7).

Suppose (3.3) admits a solution $(x(t), y(t))$. If $\frac{dx}{dt} \neq 0$, then we can write $t = \psi(x)$ with a function ψ in a neighborhood of $x_0 = x(t_0)$, and we can write

$$y(t) = y(\psi(x)) = \Psi(x), \tag{3.9}$$

as far as $\frac{dx}{dt} \neq 0$. Then the function Ψ satisfies the functional equation (3.7).

Conversely we assume that a function Ψ is a solution of the functional equation (3.7). If the first order difference equation

$$x(t + 1) = X(x(t), \Psi(x(t))), \tag{3.10}$$

has a solution $x(t)$, then we put $y(t) = \Psi(x(t))$ and have a solution $(x(t), y(t))$ of (3.3). From [11] we see that the first order difference equation (3.10) has an analytic solution.

This relation is a point of our method.

Put $\omega(t) = \Upsilon(t + 1)$ and

$$\begin{pmatrix} \chi \\ \nu \end{pmatrix} = P^{-1} \begin{pmatrix} \Upsilon \\ \omega \end{pmatrix}. \tag{3.11}$$

Then we have $\chi(t) = \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 \Upsilon(t) - \omega(t))$. Since $\Upsilon(t + n) \to 0$ and $\omega(t + n) \to 0$ as $n \to \infty$, we have $\chi(t + n) \to 0$ as $n \to \infty$.

Let $u(t)$ be a solution given in Section 2,

$$u(t) = \sum_{n=1}^{\infty} a_{1,n} \lambda^{nt} \qquad (\lambda = \lambda_1).$$

Then we can write by (3.11), since $\lambda_1 = \lambda$ and $u(t)$ is a function of λ^t,

$$x(t) = \frac{1}{\lambda_2 - \lambda}(\lambda_2 u(t) - u(t+1)) = \frac{1}{\lambda_2 - \lambda}\left(\sum_{n=1}^{\infty}(\lambda_2 a_{1,n} - a_{1,n}\lambda^n)(\lambda^t)^n\right) = \tilde{U}(\lambda^t), \tag{3.12}$$

where $\zeta = \tilde{U}(\tau)$ is a function of $\tau = \lambda^t$ and $\tilde{U}'(0) = a_{1,1} \neq 0$ and $\tilde{U}(0) = 0$. Since $\tilde{U}(\tau)$ is an open map, for any $\eta_1 > 0$ there is an $\eta_2 > 0$ such that

$$\tilde{U}(\{|\tau| < \eta_1\}) \supset \{|\zeta| < \eta_2\}.$$

Since $\chi(t+n) \to 0$ as $n \to \infty$, supposed that t belongs to a compact set K, there is an $n_0 \in \mathbb{N}$ such that for $t' \in K$

$$|\chi(t'+n)| < \eta_2 \quad (n \geqq n_0).$$

Thus there is a $\tau' = \lambda^\sigma$, such that

$$\chi(t'+n) = \tilde{U}(\tau') = \tilde{U}(\lambda^\sigma). \tag{3.13}$$

Since $\tilde{U}'(0) = a_{1,1} \neq 0$, using the theorem on implicit function we have a $\tilde{U}^{-1}$ such that

$$\lambda^\sigma = \tilde{U}^{-1}(\chi(t'+n)).$$

Put $t = t' + n$, then $\lambda^\sigma = \tilde{U}^{-1}(\chi(t))$, and we write

$$\sigma = \log_\lambda \tilde{U}^{-1}(\chi(t)) = \ell(t). \tag{3.14}$$

When there is a solution $\chi(t)$ of (3.3), from (3.10), (3.12-3.13) we have

$$\begin{aligned}
\chi(t+1) &= X(\chi(t), \Psi(\chi(t))) \\
&= X(\tilde{U}(\lambda^\sigma), \Psi(\tilde{U}(\lambda^\sigma))) \\
&= X(x(\sigma), \Psi(x(\sigma))) \\
&= x(\sigma+1) = \tilde{U}(\lambda^{\sigma+1}).
\end{aligned}$$

Hence

$$\sigma + 1 = \ell(t+1), \ \ \ell(t) + 1 = \ell(t+1).$$

If we put $\pi(t) = \ell(t) - t$, then we obtain $\pi(t+1) = \ell(t+1) - (t+1) = \ell(t) - t = \pi(t)$, and we can write as

$$\ell(t) = t + \pi(t), \tag{3.15}$$

where $\pi(t)$ is defined for a compact set K with $\Re[t]$ sufficiently large. Furthermore we can continue the $\pi(t)$ analytically as a periodic function with the period 1. Thus we have

$$\sigma = t + \pi(t).$$

From (3.13) and (3.12), $\chi(t)$ can be written as

$$\chi(t) = \tilde{U}(\lambda^{t+\pi(t)}) = x(t+\pi(t)) = \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 u(t+\pi(t)) - u(t+1+\pi(t))).$$

We have following equations, making use of the equation (3.11)

$$\begin{aligned}\Upsilon(t) &= \chi(t)+\nu(t)\\ &= \chi(t)+\Psi(\chi(t))\\ &= x(t+\pi(t))+\Psi(x(t+\pi(t)))\\ &= \frac{1}{\lambda_2-\lambda_1}(\lambda_2\sum_{n=1}^{\infty}a_{1,n}\lambda^{n(t+\pi(t))}-\sum_{n=1}^{\infty}a_{1,n}\lambda^{n(t+\pi(t)+1)})\\ &\qquad+\Psi\left(\frac{1}{\lambda_2-\lambda_1}(\lambda_2\sum_{n=1}^{\infty}a_{1,n}\lambda^{n(t+\pi(t))}-\sum_{n=1}^{\infty}a_{1,n}\lambda^{n(t+\pi(t)+1)})\right),\end{aligned}$$

where $\pi(t)$ is defined for $t\in\cup_{n\in\mathbb{Z}}(K+n)$ with a compact set K. Since K is arbitrary, we can continue $\pi(t)$ analytically to a periodic entire function with period 1, and Ψ is a solution of (3.7). By making use of the Theorem in [7], ([9]), Ψ is obtained in the form, in a neighborhood of $x=0$,

$$\Psi(x)=\sum_{n=2}^{\infty}\gamma_n x^n, \tag{3.16}$$

that is, the expansion begins with x^2. From $\chi(t+1)=X(\chi(t),\Psi(\chi(t)))$, we have

$$\chi(t+1)=\lambda_1\chi(t)+\sum_{i+j\geqq 2}c_{ij}\chi(t)^i\Psi(\chi(t))^j,$$

and

$$\frac{\chi(t+1)}{\chi(t)}=\lambda_1+\sum_{i+j\geqq 2}c_{ij}\chi(t)^{i-1}\Psi(\chi(t))^j.$$

Since $\chi(t+n)\to 0$, as $n\to+\infty$ and by (3.16),

$$\frac{\Psi(\chi(t+n))}{\chi(t+n)}\to 0,\ \frac{\chi(t+1+n)}{\chi(t+n)}\to\lambda_1,\quad \text{as}\quad n\to+\infty.$$

From $\Upsilon(t)=\chi(t)+\Psi(\chi(t))$, we have

$$\begin{aligned}\frac{\Upsilon(t+n+1)}{\Upsilon(t+n)}=\frac{\chi(t+n+1)+\Psi(\chi(t+n+1))}{\chi(t+n)+\Psi(\chi(t+n))}&=\frac{\frac{\chi(t+n+1)}{\chi(t+n)}+\frac{\Psi(\chi(t+n+1))}{\chi(t+n+1)}\cdot\frac{\chi(t+n+1)}{\chi(t+n)}}{1+\frac{\Psi(\chi(t+n))}{\chi(t+n)}}\\ &\to\lambda_1,\ \text{as } n\to+\infty.\end{aligned}$$

Conversely, if we put $\Upsilon(t)$ as (3.6), where π is an arbitrary periodic entire function, and Ψ is a solution of (3.6), then $\Upsilon(t)$ is a solution of (1.1) such that $\Upsilon(t+n)\to 0$ as $n\to+\infty$. Furthermore then we have a solution χ of (3.3) such that

$$\Upsilon(t)=\chi(t)+\Psi(\chi(t)),$$

where $\chi(t+n) \to 0$ as $n \to +\infty$. Hence we have $\frac{\Upsilon(t+1+n)}{\Upsilon(t+n)} \to \lambda_1$ as $n \to +\infty$.

Similarly in the proof of the above case, we can prove the case $m = 2$ making use of the equations in (3.4) and (3.5). □

Appendix A

We put

$$\begin{aligned} p(t) &= \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t) \\ &= g_1(t, p(t+1), p(t+2)) + g_2(t) = g_3(t, p(t+1), p(t+2)), \end{aligned}$$

in which

$$\begin{aligned} g_1(t, p(t+1), p(t+2)) &= \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) \\ &\quad - \phi(P_N(t+1), P_N(t+2)) \\ g_2(t) &= \phi(P_N(t+1), P_N(t+2)) - P_N(t). \end{aligned} \tag{1}$$

Since ϕ is holomorphic on $|w| \leqq \rho$, $|z| \leqq \rho$, using Cauchy's integral formula [1], we have

$$\frac{\partial \phi}{\partial w}(w, z) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{\phi(\xi, z)}{(\xi - w)^2} d\xi.$$

Therefore when $|w| \leqq \frac{\rho}{2}$, we have $|\xi - w| \geqq |\xi| - |w| \geqq \rho - \frac{\rho}{2} = \frac{\rho}{2}$ and

$$\left| \frac{\partial \phi}{\partial w}(w, z) \right| \leqq \frac{1}{\pi} \int_{|\xi|=\rho} \frac{|\phi(\xi, z)|}{(\frac{\rho}{2})^2} |d\xi| \leqq \frac{1}{\pi} \int_{|\xi|=\rho} \frac{K}{(\frac{\rho}{2})^2} |d\xi| = \frac{8K}{\rho}.$$

When $|z| \leqq \frac{\rho}{2}$, similarly for z we obtain

$$\left| \frac{\partial \phi}{\partial z}(w, z) \right| \leqq \frac{8K}{\rho}.$$

Hence we have

$$\left| \frac{\partial \phi}{\partial w} \right|, \left| \frac{\partial \phi}{\partial z} \right| \leqq \frac{8K}{\rho} \qquad \text{for} \quad |w|, |z| \leqq \frac{\rho}{2}.$$

Next we take A, and take η sufficiently small such that $A\eta^{N+1} < \frac{\rho}{4}$. Then for sufficiently large t, we have

$$|p(t)| \leqq A|\lambda^t|^{N+1} \leq A\eta^{N+1} < \frac{\rho}{4}.$$

And we have

$$
\begin{aligned}
|p(t+1)| &\leqq A|\lambda^{t+1}|^{N+1} = A|\lambda|^{N+1}|\lambda^t|^{N+1} < \frac{\rho}{4}, \\
|p(t+2)| &\leqq A|\lambda^{t+2}|^{N+1} = A|\lambda|^{2(N+1)}|\lambda^t|^{N+1} < \frac{\rho}{4}.
\end{aligned}
$$

Furthermore we can take t so large that $|P_N(t+1)|$, $|P_N(t+2)| < \frac{\rho}{4}$, then we obtain

$$
|w| = |p(t+1) + P_N(t+1)| \leqq \frac{\rho}{2},\ |z| = |p(t+2) + P_N(t+2)| \leqq \frac{\rho}{2}.
$$

Since

$$
\begin{aligned}
g_1(t, p(t+1), p(t+2)) &= \int_0^1 \frac{d}{dr}\phi(rp(t+1) + P_N(t+1), rp(t+2) + P_N(t+2))dr \\
&= \int_0^1 \{p(t+1)\frac{\partial \phi}{\partial w}(***) + p(t+2)\frac{\partial \phi}{\partial z}(***)\}dr,
\end{aligned}
$$

where $(***) = (rp(t+1) + P_N(t+1), rp(t+2) + P_N(t+2))$, we have

$$
\begin{aligned}
&|g_1(t, p(t+1), p(t+2))| \\
&\leqq \int_0^1 \{|p(t+1)|\left|\frac{\partial \phi}{\partial w}(***)\right| + |p(t+2)|\left|\frac{\partial \phi}{\partial z}(***)\right|\}dr, \\
&\leqq \int_0^1 \{A|\lambda^t|^{N+1}|\lambda|^{N+1} \cdot \frac{8K}{\rho} + A|\lambda^t|^{N+1}|\lambda|^{2(N+1)} \cdot \frac{8K}{\rho}\}dr \leqq \frac{16K}{\rho}A|\lambda|^{N+1} \cdot |\lambda^t|^{N+1}.
\end{aligned} \tag{2}
$$

From definition of P_N and (1), we have

$$
|g_2(t)| \leqq K_2|\lambda^t|^{N+1}, \tag{3}
$$

with a constant K_2 which depends on N. From (2) and (3), we have

$$
|T_1[p](t)| \leqq |g_1(t, p(t+1), p(t+2))| + |g_2(t)| \leqq \left(\frac{16K}{\rho}A|\lambda|^{N+1} + K_2\right)|\lambda^t|^{N+1}.
$$

If we suppose N is so large that $\frac{16K}{\rho}|\lambda|^{N+1} < \frac{1}{4}$, then we have

$$
|T_1[p](t)| \leqq \left(\frac{1}{4}A + K_2\right)|\lambda^t|^{N+1}.
$$

Furthermore we take A so large that $A > \frac{4}{3}K_2$, then

$$
|T_1[p](t)| < A|\lambda^t|^{N+1}.
$$

So we obtain that T_1 in (2.8) maps $J(A, \eta)$ into itself.

Appendix B

Suppose there is another fixed point $p^*(t) = p_N^*(t) \in J(A^*, \eta^*)$. Put

$$A_0 = \max(A, A^*), \quad \eta_0 \leqq \min(\eta, \eta^*),$$
$$u(t) = p_N(t) + P_N(t), \quad u^*(t) = p_N^*(t) + P_N(t),$$

and

$$q(t) = p_N^*(t) - p_N(t).$$

Then we have $|q(t)| \leqq 2A_0|\lambda^t|^{N+1}$. From (2.8), we have

$$\begin{aligned} q(t) &= \{\phi(p_N^*(t+1) + P_N(t+1), p_N^*(t+2) + P_N(t+2)) - P_N(t)\} \\ &\qquad - \{\phi(p_N(t+1) + P_N(t+1), p_N(t+2) + P_N(t+2)) - P_N(t)\} \\ &= \phi(q(t+1) + u_N(t+1), q(t+2) + u_N(t+2)) - \phi(u_N(t+1), u_N(t+2)) \\ &= \int_0^1 \{q(t+1)\frac{\partial \phi}{\partial w}(****) + q(t+2)\frac{\partial \phi}{\partial z}(****)\}dr \end{aligned}$$

where $(****) = (rq(t+1) + u_N(t+1), rq(t+2) + u_N(t+2))$. If η_0 is sufficiently small, then we have

$$\left|\frac{\partial \phi}{\partial w}(****)\right|, \ \left|\frac{\partial \phi}{\partial z}(****)\right| < \frac{8K_1}{\eta},$$

and we suppose N is sufficiently large such that $|\lambda|^{N+1} < \frac{\rho}{64K_1}$. Thus we have

$$\begin{aligned} |q(t)| &\leqq \int_0^1 \frac{8K_1}{\rho}(|q(t+1)| + |q(t+2)|)dr \\ &\leqq \int_0^1 \frac{8K_1}{\rho}|\lambda|^{N+1}(2A_0|\lambda^t|^{N+1} + 2A_0|\lambda^t|^{N+1})dr < \frac{1}{2}A_0|\lambda^t|^{N+1}. \end{aligned}$$

Then

$$|q(t)| = |p_N^*(t) - p_N(t)| \leqq \frac{1}{2}A_0|\lambda^t|^{N+1} = \left(\frac{1}{4}\right) \cdot 2A_0|\lambda^t|^{N+1}, \quad \text{for } t \in S(\eta_0).$$

Next we consider $q(t)$ in which $|q(t)| \leqq \frac{1}{4} \cdot 2A_0|\lambda^t|^{N+1}$ and repeat this procedure, then we have $|q(t)| \leqq (\frac{1}{4})^2 \cdot 2A_0|\lambda^t|^{N+1}$. Repeating this procedure k times we obtain

$$|p_N^*(t) - p_N(t)| < \left(\frac{1}{4}\right)^k (2A_0)|\lambda^t|^{N+1}, \quad k = 1, 2, \cdots.$$

Letting $k \to \infty$, we have

$$p_N^*(t) = p_N(t), \quad t \in S(\eta_0).$$

Thus $p_N^*(t) = p^*(t)$ and $p_N(t) = p(t)$ are holomorphic in $|\lambda^t| \leqq \min(\eta, \eta^*)$ and $p^*(t) \equiv p(t)$ in $t \in S(\eta_0)$. Hence $p_N^*(t) = p_N(t)$ can be continued analytically to $S(\eta_1)$, $\eta_1 = \max(\eta, \eta^*)$. $\square$

Appendix C

Here we will show that the solution $u_N(t)$, given by $u_N(t) = p_N(t) + P_N(t)$ does not depend on N in both cases in i). Let $p_N(t) \in J(A_N, \eta_N)$ and $p_{N+1}(t) \in J(A_{N+1}, \eta_{N+1})$ be fixed points of T_1, and

$$u_{N+1}(t) = p_{N+1}(t) + P_{N+1}(t) = p_{N+1}(t) + a_{N+1}\lambda^{(N+1)t} + P_N(t) = \tilde{p}_N(t) + P_N(t).$$

$$\begin{aligned}|\tilde{p}_N(t)| = |p_{N+1}(t) + a_{N+1}\lambda^{(N+1)t}| &\leqq A_{N+1}|\lambda^t|^{N+2} + |a_{N+1}| \cdot |\lambda^t|^{N+1} \\ &= (A_{N+1}|\lambda^t| + |a_{N+1}|)|\lambda^t|^{N+1} = A_N^*|\lambda^t|^{N+1},\end{aligned}$$

where $A^* = A_{N+1}|\lambda^t| + |a_{N+1}|$. We put $A = \max(A_N, A_N^*)$. by uniqueness of fixed point, $\tilde{p}_N(t) = p_N(t)$ for $t \in S(\eta_N) \cap S(\eta_{N+1})$. Thus

$$u_{N+1}(t) = u_N(t) \quad \text{in} \quad S(\eta_N) \cap S(\eta_{N+1}).$$

By analytic prolongation [1], both of $u_N(t)$ and $u_{N+1}(t)$ are holomorphic in $S(\eta_N) \cap S(\eta_{N+1})$ and coincide there. Hence both of them are continued analytically to $S(\eta_N) \cup S(\eta_{N+1})$ and

$$u_{N+1}(t) = u_N(t) \quad \text{in} \quad S(\eta_N) \cup S(\eta_{N+1}). \ \square$$

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